

# Algorithms in Everyday Mathematics



## What is an algorithm?

An algorithm is a well-defined procedure or set of rules guaranteed to achieve a certain objective. You use an algorithm every time you follow the directions to put together a new toy, use a recipe to make cookies, or defrost something in the microwave.

In mathematics, an algorithm is a specific series of steps that will give you the correct answer every time. For example, in grade school, you and your classmates probably learned and memorized a certain algorithm for multiplying. Chances are, no one knew why it worked, but it did!

In *Everyday Mathematics*, children first learn to understand the mathematics behind the problems they solve. Then, quite often, they come up with their own unique working algorithms that prove that they “get it.” Through this process, they discover that there is more than one algorithm for computing answers to addition, subtraction, multiplication, and division problems. Having children become comfortable with algorithms is essential to their growth and development as problem solvers.

## How do children learn to use algorithms for computation?

Ideally, children should develop a variety of computational methods and the flexibility to choose the procedure that is most appropriate in a given situation. *Everyday Mathematics* includes a variety of standard computational algorithms as well as children’s invented procedures. The program leads children through three phases as they learn each mathematical operation (addition, subtraction, multiplication, and division).

## Algorithm Invention

In the early phases of learning an operation, children are encouraged to invent their own methods for solving problems. This approach requires children to focus on the meaning of the operation. They learn to think and use their common sense as well as new skills and knowledge. Children who invent their own procedures:

- ◆ learn that their intuitive methods are valid and that mathematics makes sense.
- ◆ become more proficient with mental arithmetic.
- ◆ are motivated because they understand their own methods, as opposed to learning by rote.
- ◆ become skilled at representing ideas with objects, words, pictures, and symbols.
- ◆ develop persistence and confidence in dealing with challenging problems.

## Alternative Algorithms

After children have had many opportunities to experiment with their own computational strategies, they are introduced to several algorithms for each operation. Some of these algorithms may be the same or similar to the methods children have invented on their own. Others are traditional algorithms which have commonly been taught in the U.S. or simplifications of those algorithms. And others are entirely new algorithms that have significant advantages in today's technological world.

Children are encouraged to experiment with various algorithms and to become proficient with at least one.

## Demonstrating Proficiency

For each operation, the program designates one alternative algorithm as a "focus" algorithm. Focus algorithms are powerful, relatively efficient, and easy to understand and learn. They also provide common and consistent language, terminology, and support across grade levels of the curriculum.

All children are expected to learn and demonstrate proficiency with the focus algorithm for each operation. Once they can reliably use the focus algorithm, children may use it or any alternative they prefer when solving problems. The aim of this approach is to promote flexibility while ensuring that all children know at least one reliable method for each operation.

## Online Resources on Algorithms

For more information on algorithms used in *Everyday Mathematics* and details about what resources are available online, see pages 60 and 61.

## ▷ Addition Algorithms

The following are just a few of the possible algorithms for adding whole numbers.

### Focus Algorithm:

#### Partial-Sums Addition

You can add two numbers by calculating partial sums, working one place-value column at a time, and then adding all the sums to find the total.

#### Example: Partial-Sums Addition

	268
	+ 483
Add the hundreds ( $200 + 400$ ).	600
Add the tens ( $60 + 80$ ).	140
Add the ones ( $8 + 3$ ).	+ 11
Add the partial sums ( $600 + 140 + 11$ ).	751

To add using the column-addition algorithm, draw vertical lines to separate the ones, tens, hundreds, and so on. Add the digits in each column, and then adjust the results.

Add the digits in each column.

hundreds	tens	ones
2	6	8
+ 4	8	3
6	14	11

hundreds	tens	ones
2	6	8
+ 4	8	3
7	4	11

hundreds	tens	ones
2	6	8
+ 4	8	3
7	5	1

If you add a number to one part of a sum and subtract the same number from the other part, the result remains the same. For example, consider:

$$8 + 7 = 15$$

$$(8 + 2) + (7 - 2) = 10 + 5 = 15$$

Rename the first number and then the second.

$$\begin{array}{r} 268 \\ + 483 \\ \hline \end{array}$$

$$\begin{array}{r} 270 \\ + 481 \\ \hline \end{array}$$

$$\begin{array}{r} 300 \\ + 451 \\ \hline 751 \end{array}$$

Add 2.      Add 30.  
 Subtract 2.      Subtract 30.

Subtract 7. Subtract 10.

$$\begin{array}{r} 268 \\ + 483 \\ \hline \end{array}$$

Subtract 7. Subtract 10.

$$\begin{array}{r} 261 \\ + 490 \\ \hline \end{array}$$

Subtract 7. Subtract 10.

$$\begin{array}{r} 251 \\ + 500 \\ \hline \end{array}$$

Subtract 7. Subtract 10.

$$\begin{array}{r} 251 \\ + 500 \\ \hline 751 \end{array}$$

### U.S. Traditional Addition

Using this algorithm, add one place-value column at a time. If the sum is two digits, write one of the digits above the addends; this is sometimes called "carrying."

#### Example: U.S. Traditional Addition

Start with the 1s. Write 1 in the 1s place below the line and 1 above the numbers in the 10s place.

$$\begin{array}{r} 1 \\ 268 \\ + 483 \\ \hline 1 \end{array}$$

Add the 10s. Write 5 in the 10s place below the line and 1 above the numbers in the 100s place.

$$\begin{array}{r} 11 \\ 268 \\ + 483 \\ \hline 51 \end{array}$$

Then add the 100s. Write 7 in the 100s place below the line.

$$\begin{array}{r} 11 \\ 268 \\ + 483 \\ \hline 751 \end{array}$$

### ▷ Subtraction Algorithms

This section presents several subtraction algorithms.

#### Focus Algorithm:

##### Trade-First Subtraction

This algorithm is similar to the traditional U.S. algorithm except that all the trading is done before the subtraction, allowing children to concentrate on one thing at a time.

#### Example: Trade-First Subtraction

Look at the 1s place. You cannot remove 6 ones from 2 ones.

hundreds	tens	ones
9	3	2
- 3	5	6

So trade 1 ten for 10 ones. Now look at the 10s place. You cannot remove 5 tens from 2 tens.

hundreds	tens	ones
9	2	12
- 3	<del>5</del>	<del>6</del>

So trade 1 hundred for 10 tens. Now subtract in each column.

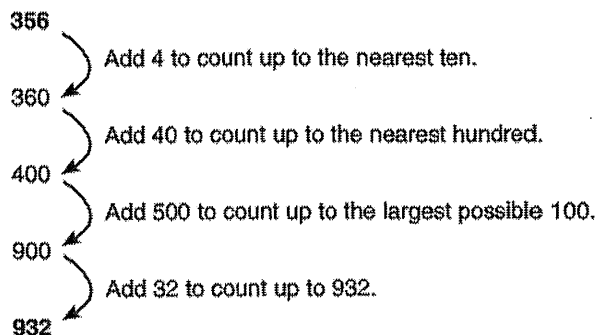
hundreds	tens	ones
8	12	12
<del>9</del>	<del>2</del>	<del>12</del>
- 3	5	6
5	7	6

### Counting Up

To subtract using the counting-up algorithm, start with the number you are subtracting (the subtrahend), and "count up" to the number you are subtracting from (the minuend) in stages. Keep track of the amounts you count up at each stage. When you are finished, find the sum of the amounts.

#### Example: Counting Up

To find  $932 - 356$ , start with 356 and count up to 932.



Now find the sum of the numbers you added.

$$\begin{array}{r} 4 \\ 40 \\ 500 \\ + 32 \\ \hline 576 \end{array}$$

So,  $932 - 356 = 576$ .

### Partial-Differences Subtraction

The partial-differences subtraction algorithm is a fairly unusual method, but one that appeals to some children.

The procedure is fairly simple: Write partial differences for each place, record them, and then add them to find the total difference. Sometimes some of the partial differences may be negative.

#### Example: Partial-Differences Subtraction

		932
		<u>- 356</u>
Subtract 100s.	$900 - 300$	600
Subtract 10s.	$30 - 50$	- 20
Subtract 1s.	$2 - 6$	- 4
Add the partial differences.		<u>576</u>

**U.S. Traditional Subtraction**

To subtract using this algorithm, start with the 1s. Regroup as necessary and then subtract the 1s. Children move from right to left as they regroup and subtract in each column.

**Example: U.S. Traditional Subtraction**

Start with the 1s. Regroup.  
Trade 1 ten for 10 ones.  
Subtract the 1s.

$$\begin{array}{r} 212 \\ 932 \\ - 356 \\ \hline 6 \end{array}$$

Go to the 10s. Regroup.  
Trade 1 hundred for 10 tens.  
Subtract the 10s.

$$\begin{array}{r} 12 \\ 8212 \\ 932 \\ - 356 \\ \hline 76 \end{array}$$

Go to the 100s. You don't need to regroup. Subtract the 100s.

$$\begin{array}{r} 12 \\ 8212 \\ 932 \\ - 356 \\ \hline 576 \end{array}$$

**► Multiplication Algorithms**

Children's experiences with addition and subtraction algorithms can help them with multiplication algorithms, such as the partial-product multiplication algorithm. This algorithm and others are discussed in this section.

**Focus Algorithm: Partial Products**

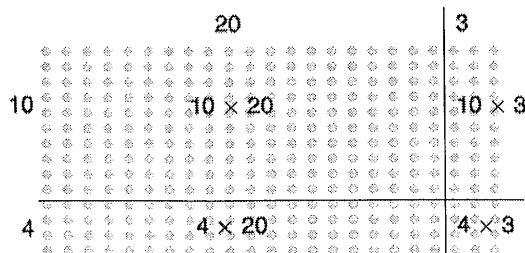
To use the partial-products algorithm, think of each factor as the sum of ones, tens, hundreds, and so on. Then multiply each part of one sum by each part of the other, and add the results.

Rectangular arrays can be used to demonstrate visually how the partial-products algorithm works. The product  $14 \times 23$  is the number of dots in a 14-by-23 array.

**Example: Partial Products**

To find  $67 \times 53$ , think of 67 as 60 + 7 and 53 as 50 + 3. Multiply each part of 67 by each part of 53. Then add the results.

$$\begin{array}{r} 67 \\ \times 53 \\ \hline 50 \times 60 \rightarrow 3000 \\ 50 \times 7 \rightarrow 350 \\ 3 \times 60 \rightarrow 180 \\ 3 \times 7 \rightarrow 21 \\ \hline \text{Add the results.} \quad 3,551 \end{array}$$



$$\begin{aligned} 14 \times 23 &= (10 + 4) \times (20 + 3) \\ &= (10 \times 20) + (10 \times 3) + (4 \times 20) + (4 \times 3) \\ &= 200 + 30 + 80 + 12 \\ &= 322 \end{aligned}$$

### Modified Repeated Addition

Many children are taught to think of whole-number multiplication as repeated addition. However, using repeated addition as a computation method is inefficient for anything but small numbers. For example, it would be extremely tedious to add fifty-three 67s in order to compute  $67 \times 53$ . Using a modified repeated addition algorithm, in which multiples of 10, 100, and so on, are grouped together, can simplify the process.

### Example: Modified Repeated Addition

Think of  $53 \times 67$  as  
fifty 67s plus three 67s.  
Since ten 67s is 670,  
fifty 67s is five 670s.

So,  $53 \times 67$  is five 670s  
plus three 67s.

$$\begin{array}{r} 67 \\ \times 53 \\ \hline 670 \\ 670 \\ 670 \\ 670 \\ 670 \\ 67 \\ 67 \\ 67 \\ \hline 3,551 \end{array}$$

### Lattice Multiplication

The following example shows how the method is used to find  $67 \times 53$ .

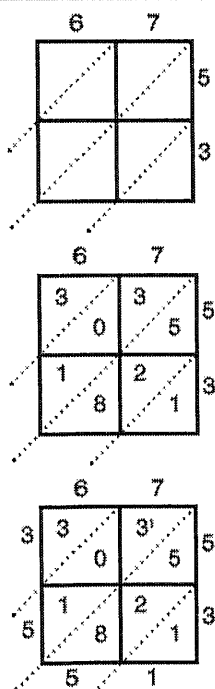
### Example: Lattice Multiplication

Follow these steps to find  $67 \times 53$ .

- Draw a 2-by-2 lattice, and write one factor along the top of the lattice and the other along the right. (Use a larger lattice to multiply numbers with more digits.)
- Draw diagonals from the upper-right corner of each box, extending beyond the lattice.
- Multiply each digit in one factor by each digit in the other. Write each product in the cell where the corresponding row and column meet. Write the tens digit of the product above the diagonal and the ones digit below the diagonal. For example, since  $6 \times 5 = 30$ , write 30 in the upper-left box with the 3 above the diagonal and the 0 below the diagonal.
- Starting with the lower-right diagonal, add the numbers inside the lattice along each diagonal. If the sum along a diagonal is greater than 9, carry the tens digit to the next diagonal.

The first diagonal contains only 1, so the sum is 1. The sum on the second diagonal is  $5 + 2 + 8 = 15$ . Write only the 5, and carry the 1 to the next column. The sum along the third diagonal is then  $1 + 3 + 0 + 1$ , or 5. The sum on the fourth diagonal is 3.

- Read the product from the upper left to the lower right. The product is 3,551.



**U.S. Traditional Multiplication**

Multiply from right to left using this algorithm. Writing a 2 above the tens place as shown below is called "carrying" the 2. The numbers that are carried are called "carries."

**Example: U.S. Traditional Multiplication**

Multiply 67 by the 3 in 53 as if the problem were  $3 \times 67$ .

Multiply 67 by the 5 in 53 as if the problem were  $5 \times 67$ . The 5 in 53 stands for 5 tens, so write this partial product one place to the left. Write a 0 in the 1s place to show you are multiplying by tens. Write the new carries above the old carries.

Add the two partial products to get the final answer.

$$\begin{array}{r}
 2 \\
 67 \\
 \times 53 \\
 \hline
 201 \\
 3350 \\
 \hline
 3,551
 \end{array}$$

**▷ Division Algorithms**

One type of division situation involves making as many equal-size groups as possible from a collection of objects. More generally,  $a \div b$  can be interpreted as "How many  $b$ s are in  $a$ ?" This idea forms the basis for the division algorithms presented in this section.

**Focus Algorithm: Partial Quotients**

This algorithm uses a series of estimates of how many  $b$ s are in  $a$ .

**Example: Partial Quotients**

**Estimate the number of 12s in 158.**

You might begin with multiples of 10 because they are simple to work with. There are at least ten 12s in 158 ( $10 \times 12 = 120$ ), but there are fewer than twenty 12s ( $20 \times 12 = 240$ ). Record 10 as a first estimate, and subtract ten 12s from 158, leaving 38.

$$\begin{array}{r|l}
 12 \overline{)158} & \\
 \underline{-120} & 10 \text{ first guess} \\
 38 & \\
 \underline{-36} & 3 \text{ second guess} \\
 2 & 13 \text{ sum of guesses}
 \end{array}$$

**Now estimate the number of 12s in 38.**

There are more than three ( $3 \times 12 = 36$ ) but fewer than four ( $4 \times 12 = 48$ ). Record 3 as the next estimate, and subtract three 12s from 38, leaving 2.

$$158 \div 12 \longrightarrow 13 \text{ R}2$$

Since 2 is less than 12, you can stop estimating. The final result is the sum of the estimates ( $10 + 3 = 13$ ) plus what is left over (the remainder of 2).



## Column Division

Column division is a simplification of the traditional long division algorithm you probably learned in school, but it is easier to learn. To use the method, you draw vertical lines separating the digits of the divisor and work one place-value column at a time.

### Example: Column Division

To find  $683 \div 5$ , imagine sharing \$683 among 5 people. Think about having 6 hundred-dollar bills, 8 ten-dollar bills, and 3 one-dollar bills.

First, divide up the hundred-dollar bills. Each person gets one, and there is one left over.

1	8	3
5)6		
-5		
1		

Trade the leftover hundred-dollar bill for 10 ten-dollar bills. Now you have a total of 18 ten-dollar bills. Each person gets 3, and there are 3 left over.

1	3	3
5)6	<del>8</del>	
-5	18	
<del>1</del>	-15	
	3	

Trade the 3 leftover ten-dollar bills for 30 one-dollar bills. You now have a total of 33 one-dollar bills. Each person gets 6, and there are 3 left over.

1	3	6
5)6	<del>8</del>	<del>8</del>
-5	18	33
<del>1</del>	-15	-30
	<del>3</del>	3

So, when you divide \$683 among 5 people, each person gets \$136, and there are \$3 left over. So,  $683 \div 5 \rightarrow 136 \text{ R}3$ .

### U.S. Traditional Long Division

This algorithm is similar to column division. To use this method, write the divisor to the left of the dividend. The quotient will go on top of the dividend.

#### Example: U.S. Traditional Long Division

To begin, think about sharing \$683 among 5 people.

Share the [\$100]s.

$$\begin{array}{r} 1 \\ 5 \overline{)683} \\ \underline{-5} \phantom{00} \\ 1 \phantom{00} \end{array}$$

Trade 1 [\$100]s for 10 [\$10]s.  
That makes 18 [\$10]s in all.

$$\begin{array}{r} 1 \\ 5 \overline{)683} \\ \underline{-5} \phantom{00} \\ 18 \phantom{00} \end{array}$$

Share the [\$10]s.

$$\begin{array}{r} 13 \\ 5 \overline{)683} \\ \underline{-5} \phantom{00} \\ 18 \phantom{00} \\ \underline{-15} \phantom{00} \\ 3 \phantom{00} \end{array}$$

Trade 3 [\$10]s for 30 [\$1]s.  
That makes 33 [\$1]s in all.

$$\begin{array}{r} 13 \\ 5 \overline{)683} \\ \underline{-5} \phantom{00} \\ 18 \phantom{00} \\ \underline{-15} \phantom{00} \\ 33 \phantom{00} \end{array}$$

Share the [\$1]s.

$$\begin{array}{r} 136 \\ 5 \overline{)683} \\ \underline{-5} \phantom{00} \\ 18 \phantom{00} \\ \underline{-15} \phantom{00} \\ 33 \phantom{00} \\ \underline{-30} \phantom{00} \\ 3 \phantom{00} \end{array}$$

Each person gets \$136; \$3 is left over.

$$683 \div 5 \rightarrow 136 \text{ R}3$$